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On the field of definition for modularity of CM elliptic curves

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1 Introduction

Let E be a CM elliptic curve defined over an algebraic number field $F \subseteq \mathbb{C}$ whose \mathbb{Q} -algebra of endomorphisms defined over $\overline{\mathbb{Q}}$, denoted by $\text{End}_{\overline{\mathbb{Q}}}^0(E)$, is isomorphic to an imaginary quadratic field $K \subseteq \mathbb{C}$. We take an integral ideal \mathfrak{m} in K and denote by $I_K(\mathfrak{m})$ the group of fractional ideals in K prime to \mathfrak{m} . We consider a homomorphism $\lambda : I_K(\mathfrak{m}) \rightarrow \mathbb{C}^\times$ such that (i) $\lambda((\alpha)) = \alpha$ for any $\alpha \in K^\times$ s.t. $\alpha \equiv 1 \pmod{\mathfrak{m}}$; (ii) λ is primitive, i.e. there is no proper divisor \mathfrak{n} of \mathfrak{m} such that λ has a extension $\tilde{\lambda}$ to $I_K(\mathfrak{n})$ with the property: $\tilde{\lambda}((\alpha)) = \alpha$ for any $\alpha \in K^\times$ s.t. $\alpha \equiv 1 \pmod{\mathfrak{n}}$. Then we put

$$f_\lambda(z) := \sum_{\substack{\mathfrak{a} \in I_K(\mathfrak{m}) \\ \mathfrak{a}: \text{integral}}} \lambda(\mathfrak{a}) e^{2\pi i N(\mathfrak{a})z} \quad (z \in \mathfrak{H}, \text{ the complex upper plane}),$$

where $N(\mathfrak{a})$ denotes the absolute norm of an ideal \mathfrak{a} . Let $-D$ be the discriminant of K and put $N := DN(\mathfrak{m})$. We define a Dirichlet character $\varepsilon : (\mathbb{Z}/N\mathbb{Z})^\times \rightarrow \mathbb{C}^\times$ by

$$\overline{a} \mapsto \left(\frac{-D}{a} \right) \frac{\lambda((a))}{a} \quad (a \in \mathbb{Z}, (a, N) = 1),$$

where if $a = p_1^{e_1} \cdots p_r^{e_r}$ is the factorization of a into prime factors,

$$\left(\frac{-D}{a} \right) = \prod_{i=1}^r \left(\frac{-D}{p_i} \right)^{e_i}, \quad \left(\frac{-D}{p_i} \right) = \begin{cases} 1 & \text{if } p_i \text{ splits in } K/\mathbb{Q} \\ -1 & \text{if } p_i \text{ is inert in } K/\mathbb{Q}. \end{cases}$$

By Hecke-Shimura, we have the following:

Fact 1. f_λ is a normalized newform of weight two on $\Gamma_1(N)$ and ε is the Nebentypus of f_λ .

By the Eichler-Shimura theory, for any normalized newform f of weight two on $\Gamma_1(M)$, we can associate the abelian variety J_f defined over \mathbb{Q} which is a \mathbb{Q} -simple factor of $J_1(M)$, the jacobian variety of the modular curve $X_1(M)$. Shimura proved the following (see Proposition 1.6 and Remark 1.7 in [5]):

Fact 2. $\text{Hom}_{\overline{\mathbb{Q}}}(E, J_f) \neq \{0\}$ if and only if there exists an above λ such that $f = f_\lambda$, where $\text{Hom}_{\overline{\mathbb{Q}}}(E, J_f)$ denotes the additive group of homomorphisms from E to J_f defined over $\overline{\mathbb{Q}}$.

For any imaginary quadratic field K , if we take an integral ideal \mathfrak{m}_0 in K such that

$$\zeta \in K, \quad \zeta \text{ is a root of unity, } \quad \zeta \equiv 1 \pmod{\mathfrak{m}_0} \implies \zeta = 1$$

holds (we can always do so), there exists a homomorphism $\lambda : I_K(\mathfrak{m}_0) \rightarrow \mathbb{C}^\times$ satisfying the condition (i). Replacing \mathfrak{m}_0 by the minimal divisor \mathfrak{m} of \mathfrak{m}_0 such that λ has an extension $\tilde{\lambda}$ to $I_K(\mathfrak{m})$ and $\tilde{\lambda}$ has also the property (i), we may assume that λ is primitive. Therefore we have

Fact 3. For any CM elliptic curve E defined over an algebraic number field F , there exists a newform f such that a non-zero homomorphism $\varphi : E \rightarrow J_f$ defined over $\overline{\mathbb{Q}}$ exists, that is, E is modular over $\overline{\mathbb{Q}}$.

In this paper we will consider the following questions.

Question 1. Let E/F be as above. Under what condition does there exist a newform f such that a non-zero homomorphism $\varphi : E \rightarrow J_f$ defined over F exists, that is, when is E modular over F ?

Question 2. Assume that E/F is modular over F . Therefore there exists a newform f with $\text{Hom}_F(E, J_f) \neq \{0\}$. Then, how large is $\text{Hom}_F(E, J_f)$? In other words, decide the multiplicity of E as F -simple factor of J_f .

2 Preliminaries

Let E/F , K , $\lambda : I_K(\mathfrak{m}) \rightarrow \mathbb{C}^\times$, and $f = f_\lambda$ be as in the introduction. Let $f = \sum_{m \geq 1} a_m q^m$ ($q = e^{2\pi iz}$) be the Fourier expansion at $i\infty$ and put $H := \mathbb{Q}(a_m | m \geq 1)$ ($\subseteq \mathbb{C}$). Let n be the dimension of J_f , then H is an algebraic number field with $[H : \mathbb{Q}] = n$. A \mathbb{Q} -algebra isomorphism $\theta : H \xrightarrow{\sim} \text{End}_{\mathbb{Q}}^0(J_f) = \text{End}_{\mathbb{Q}}(J_f) \otimes_{\mathbb{Z}} \mathbb{Q}$ is defined by

$a_m \mapsto$ the endomorphism of J_f induced by the m -th Hecke operator w.r.t. $\Gamma_1(N)$ ($m = 1, 2, \dots$). In [3] Shimura proved that J_f is isogenous to $E^n = E \times \dots \times E$ (n terms) over $\overline{\mathbb{Q}}$, expressed by $J_f \sim_{\overline{\mathbb{Q}}} E^n$. So we have $\text{End}_{\overline{\mathbb{Q}}}^0(J_f) \cong M_n(K)$, the algebra

of $n \times n$ -matrices with entries in K . Let Z be the center of $\text{End}_{\mathbb{Q}}^0(J_f)$. Then we have $Z \cong K$. We denote by T the sub \mathbb{Q} -algebra of $\text{End}_{\mathbb{Q}}^0(J_f)$ generated by Z and $\theta(H)$. Shimura used the following facts in the proof of Proposition 1.6 in [5] and we state them as a lemma without proof.

Lemma 2.1. (1) $Z \cap \theta(H) = \mathbb{Q}$. Especially this implies that $\dim_H T = 2$.
 (2) $\text{End}_K^0(J_f) = T$.

Therefore, as for the structure of T , we have the possibility of the following two cases:

- Case 1 :** T is isomorphic to an algebraic number field with degree $2n$ (over \mathbb{Q})
 $\iff K \not\subseteq H$;
Case 2 : $T \cong H \oplus H \iff K \subseteq H$.

Let $F' = \langle F, K \rangle$ be the subfield of \mathbb{C} generated by F and K . It is well known that $\text{End}_{\mathbb{Q}}^0(E) = \text{End}_{F'}^0(E) (\cong K)$. We put $\mathcal{M} := \text{Hom}_{\overline{\mathbb{Q}}}(E, J_f) \otimes_{\mathbb{Z}} \mathbb{Q}$. Then the absolute Galois group $\text{Gal}(\overline{\mathbb{Q}}/F)$ over F acts on \mathcal{M} by the action on coefficients of homomorphisms. If we know the structure of \mathcal{M} as Galois module, we will be able to answer Questions 1 and 2. Therefore our purpose in this paper is to determine the structure of \mathcal{M} as $\text{Gal}(\overline{\mathbb{Q}}/F)$ -module. On the other hand we have the following.

Lemma 2.2. $\text{Hom}_{F'}(E, J_f) \neq \{0\} \iff \text{Hom}_F(E, J_f) \neq \{0\}$.

By this lemma, for answer to Question 1, it is enough to study the structure of \mathcal{M} as $\text{Gal}(\overline{\mathbb{Q}}/F')$ -module. But, for answer to Question 2, this does not seem to be enough. Nevertheless, as we will see later, under assumption $\text{Hom}_{F'}(E, J_f) \neq \{0\}$ the structure of \mathcal{M} as $\text{Gal}(\overline{\mathbb{Q}}/F)$ -module can be easily recovered from that of \mathcal{M} as $\text{Gal}(\overline{\mathbb{Q}}/F')$ -module. Therefore, in the following we will study the $\text{Gal}(\overline{\mathbb{Q}}/F')$ -module structure.

By composition of homomorphisms, \mathcal{M} has the structure of left T - and right K -module:

$$T = \text{End}_K^0(J_f) \curvearrowright \mathcal{M} \curvearrowleft \text{End}_{F'}^0(E) \cong K.$$

As $J_f \sim_{\overline{\mathbb{Q}}} E^n$, we have

$$\mathcal{M} \cong \text{Hom}_{\overline{\mathbb{Q}}}(E, E^n) \otimes_{\mathbb{Z}} \mathbb{Q} \cong K^{\oplus n}$$

as \mathbb{Q} -vector space. In particular we have $\dim_{\mathbb{Q}} \mathcal{M} = n \times \dim_{\mathbb{Q}} K = 2n$. On the other hand $H \xrightarrow{\sim} \theta(H) \subseteq T$, we can view \mathcal{M} as H -vector space. Since $[H : \mathbb{Q}] \times \dim_H \mathcal{M} = \dim_{\mathbb{Q}} \mathcal{M} = 2n$, we have $\dim_H \mathcal{M} = 2$.

Proposition 2.3. \mathcal{M} is a free left T -module of rank 1.

Let ℓ be a prime number and put

$$V_\ell(E) := T_\ell(E) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell, \quad V_\ell(J_f) := T_\ell(J_f) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell, \quad \mathcal{M}_\ell := \mathcal{M} \otimes_{\mathbb{Q}} \mathbb{Q}_\ell,$$

where $T_\ell(E)$ and $T_\ell(J_f)$ are Tate modules. We can consider the following actions:

- $\text{Gal}(\overline{\mathbb{Q}}/F) \curvearrowright \mathcal{M}_\ell \otimes_{K \otimes_{\mathbb{Q}} \mathbb{Q}_\ell} V_\ell(E)$ by diagonal;
- $H \hookrightarrow T \curvearrowright \mathcal{M}_\ell \otimes_{K \otimes_{\mathbb{Q}} \mathbb{Q}_\ell} V_\ell(E)$ by the action on \mathcal{M} .

We define a homomorphism $\nu : \mathcal{M}_\ell \otimes_{K \otimes_{\mathbb{Q}} \mathbb{Q}_\ell} V_\ell(E) \longrightarrow V_\ell(J_f)$ by

$$(\varphi \otimes a) \otimes x \longmapsto a\varphi(x).$$

Proposition 2.4. *ν is an isomorphism of (left) $H \otimes_{\mathbb{Q}} \mathbb{Q}_\ell [\text{Gal}(\overline{\mathbb{Q}}/F)]$ -modules and is also an isomorphism of (left) $T \otimes_{\mathbb{Q}} \mathbb{Q}_\ell [\text{Gal}(\overline{\mathbb{Q}}/F')]$ -modules, where $H \otimes_{\mathbb{Q}} \mathbb{Q}_\ell [\text{Gal}(\overline{\mathbb{Q}}/F)]$ (resp. $T \otimes_{\mathbb{Q}} \mathbb{Q}_\ell [\text{Gal}(\overline{\mathbb{Q}}/F')]$) denotes the group algebra of $\text{Gal}(\overline{\mathbb{Q}}/F)$ (resp. $\text{Gal}(\overline{\mathbb{Q}}/F')$) over $H \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$ (resp. $T \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$).*

3 The action of $\text{Gal}(\overline{\mathbb{Q}}/F')$ on $\mathcal{M}_\ell \otimes_{K \otimes_{\mathbb{Q}} \mathbb{Q}_\ell} V_\ell(E)$

We review the known results about the structure of $V_\ell(E)$ as $\text{Gal}(\overline{\mathbb{Q}}/F')$ -module. By changing $\iota : K \xrightarrow{\sim} \text{End}_{F'}^0(E)$ if necessary, we may assume that the CM-type of (E, ι) is $(K; \{id\})$. Then there exists a lattice \mathfrak{a} of K such that the following commutative diagram holds:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathfrak{a} & \longrightarrow & K_{\mathbb{R}} & \longrightarrow & K_{\mathbb{R}}/\mathfrak{a} \longrightarrow 0 \quad (\text{exact}) \\ & & \downarrow & & \downarrow q & & \downarrow r \\ 0 & \longrightarrow & q(\mathfrak{a}) & \longrightarrow & \mathbb{C} & \xrightarrow{\xi} & E(\mathbb{C}) \longrightarrow 0 \quad (\text{exact}), \end{array}$$

where $K_{\mathbb{R}} := K \otimes_{\mathbb{Q}} \mathbb{R}$ and $q(a \otimes x) = ax$. By the theory of complex multiplication, the following is well known (see Theorem 19.8, p. 134 in [6]).

Theorem 3.1. (1) *Every point of $E(\mathbb{C})$ with finite order is F'_{ab} -rational, where F'_{ab} denotes the maximal abelian extension of F' .*

(2) *There exists a unique homomorphism $\alpha_{E/F'} : F'^{\times}_{\mathbb{A}} \longrightarrow K^{\times}$ (where $F'^{\times}_{\mathbb{A}}$ denotes the idele group of F') such that*

- $\text{Ker}(\alpha_{E/F'})$ is open in $F'^{\times}_{\mathbb{A}}$;
- For any $x \in F'^{\times}_{\mathbb{A}}$, $\alpha_{E/F'}(x) N_{F'/K}(x)^{-1} \mathfrak{a} = \mathfrak{a}$, where $N_{F'/K}$ is the norm map from $F'^{\times}_{\mathbb{A}}$ to $K^{\times}_{\mathbb{A}}$;

- For any $x \in F'_A{}^\times$, $\alpha_{E/F'}(x)\rho(\alpha_{E/F'}(x)) = N(il(x))$, where $\rho(v)$ is the complex conjugate of a complex number v and $il(x)$ is the fractional ideal of F' associated to an idele element x ;
- For any $x \in F'_A{}^\times$ and $w \in K/\mathfrak{a}$, ${}^{[x, F']}r(w) = r(\alpha_{E/F'}(x)N_{F'/K}(x)^{-1}w)$, where $[x, F']$ is the element of $\text{Gal}(F'_{ab}/F')$ corresponding to x by the reciprocity law of class field theory.

Since $V_\ell(E)$ is viewed as free $K \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$ -module of rank 1 by ι , the action of $\text{Gal}(\overline{\mathbb{Q}}/F')$ on $V_\ell(E)$ determines the homomorphism

$$\vartheta : \text{Gal}(\overline{\mathbb{Q}}/F') \longrightarrow (K \otimes_{\mathbb{Q}} \mathbb{Q}_\ell)^\times.$$

Then ϑ factors through the restriction map to F'_{ab} . So we denote by $\overline{\vartheta}$ the induced map from $\text{Gal}(F'_{ab}/F')$ to $(K \otimes_{\mathbb{Q}} \mathbb{Q}_\ell)^\times$ and by $\tilde{\vartheta}$ the composition of the reciprocity map for F' and $\overline{\vartheta}$. Thus we have the following commutative diagram:

$$\begin{array}{ccc} \text{Gal}(\overline{\mathbb{Q}}/F') & \xrightarrow{\vartheta} & (K \otimes_{\mathbb{Q}} \mathbb{Q}_\ell)^\times \\ \text{restriction} \searrow & & \nearrow \overline{\vartheta} \\ & \text{Gal}(F'_{ab}/F') & \\ \text{reciprocity} \nearrow & & \nearrow \tilde{\vartheta} \\ & F'_A{}^\times & \end{array}$$

Then Theorem 3.1 implies the following:

Corollary 3.2. For any $x \in F'_A{}^\times$, $\tilde{\vartheta}(x) = (\alpha_{E/F'}(x)N_{F'/K}(x)^{-1})_\ell$, where $(\)_\ell$ denotes the ℓ -component.

By Proposition 2.3, the action of $\text{Gal}(\overline{\mathbb{Q}}/F')$ on \mathcal{M} determines the homomorphism

$$\chi : \text{Gal}(\overline{\mathbb{Q}}/F') \longrightarrow T^\times.$$

Let χ_ℓ be the composition of χ and the canonical map $T^\times \longrightarrow (T \otimes_{\mathbb{Q}} \mathbb{Q}_\ell)^\times$, then χ_ℓ corresponds to the action of $\text{Gal}(\overline{\mathbb{Q}}/F')$ on \mathcal{M}_ℓ . In other words, taking a basis η of \mathcal{M} over T , we have ${}^\sigma\eta = \chi(\sigma) \circ \eta$ for any $\sigma \in \text{Gal}(\overline{\mathbb{Q}}/F')$.

Firstly we consider Case 1. Since K acts T -linearly on \mathcal{M} , we can take a \mathbb{Q} -algebra isomorphism $\kappa : K \xrightarrow{\sim} Z \subseteq T$ such that $\eta \circ \iota(a) = \kappa(a) \circ \eta$ for any $a \in K$, denoted by $\eta a = a\eta$ for short. We take a basis v of $V_\ell(E)$ over $K \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$. Then $\omega := \eta \otimes v$ becomes a free basis of $\mathcal{M}_\ell \otimes_{K \otimes_{\mathbb{Q}} \mathbb{Q}_\ell} V_\ell(E)$ over $T \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$ and it holds that

$$\begin{aligned} {}^\sigma\omega &= {}^\sigma\eta \otimes {}^\sigma v = (\chi_\ell(\sigma) \circ \eta) \otimes (\vartheta(\sigma)v) = (\chi_\ell(\sigma) \circ \eta \circ (\iota \otimes 1)(\vartheta(\sigma))) \otimes v \\ &= (\chi_\ell(\sigma)\vartheta(\sigma)\eta) \otimes v = \chi_\ell(\sigma)\vartheta(\sigma)\omega \end{aligned}$$

for any $\sigma \in \text{Gal}(\overline{\mathbb{Q}}/F')$.

Next we consider Case 2. $\sqrt{-D} \in K$ acts T -linearly on \mathcal{M} , so there exists some $t \in T$ such that $\eta' \circ \iota(\sqrt{-D}) = t \circ \eta'$ for any $\eta' \in \mathcal{M}$. We will show that $t \in Z$ (one should note that in Case 2, T has two \mathbb{Q} -subalgebras isomorphic to K , so it is not trivial that $t \in Z$). For any $\varphi \in \text{End}_{\overline{\mathbb{Q}}}^0(J_f)$ and $\eta' \in \mathcal{M}$, we have

$$(\varphi \circ t) \circ \eta' = \varphi \circ (t \circ \eta') = \varphi \circ (\eta' \circ \iota(\sqrt{-D})) = (\varphi \circ \eta') \circ \iota(\sqrt{-D}) = t \circ (\varphi \circ \eta') = (t \circ \varphi) \circ \eta',$$

therefore $t \circ \varphi = \varphi \circ t$ in $\text{End}_{\overline{\mathbb{Q}}}^0(J_f)$, hence $t \in Z$. This concludes that similarly with Case 1, there exists a \mathbb{Q} -algebra isomorphism $\kappa : K \xrightarrow{\sim} Z \subseteq T$ with the same property. Let $\gamma_1 : K \hookrightarrow H$ be the map induced by the inclusion $K \subseteq H$ and $\gamma_2 : K \hookrightarrow H$ be the other homomorphism. We define an isomorphism of \mathbb{Q} -algebras $\varepsilon : T \xrightarrow{\sim} H \oplus H$ by

$$z \in Z \mapsto (\gamma_1(\kappa^{-1}(z)), \gamma_2(\kappa^{-1}(z))), \quad \theta(a) \in \theta(H) \mapsto (a, a).$$

For $k = 1, 2$, we set

$$\chi_\ell^{(k)} : \text{Gal}(\overline{\mathbb{Q}}/F') \xrightarrow{\chi_\ell} (T \otimes_{\mathbb{Q}} \mathbb{Q}_\ell)^\times \xrightarrow[\varepsilon \otimes 1]{\sim} (H \otimes_{\mathbb{Q}} \mathbb{Q}_\ell)^\times \oplus (H \otimes_{\mathbb{Q}} \mathbb{Q}_\ell)^\times \xrightarrow[k\text{-th component}]{\text{projection to}} (H \otimes_{\mathbb{Q}} \mathbb{Q}_\ell)^\times.$$

These arguments imply the following:

Proposition 3.3. *Let the notations be as above. We regard $K \otimes_{\mathbb{Q}} \mathbb{Q}_\ell \subseteq T \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$ by injection $\kappa \otimes 1$.*

(1) *In Case 1, it holds that for any $\sigma \in \text{Gal}(\overline{\mathbb{Q}}/F')$,*

$$\sigma \omega = \chi_\ell(\sigma) \vartheta(\sigma) \omega.$$

(2) *In Case 2, identifying $T \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$ with $(H \otimes_{\mathbb{Q}} \mathbb{Q}_\ell)^{\oplus 2}$ by $\varepsilon \otimes 1$, it holds that for any $\sigma \in \text{Gal}(\overline{\mathbb{Q}}/F')$,*

$$\sigma \omega = (\chi_\ell^{(1)}(\sigma) \gamma_1(\vartheta(\sigma)), \chi_\ell^{(2)}(\sigma) \gamma_2(\vartheta(\sigma))) \omega,$$

where we denote $\gamma_k \otimes 1 : K \otimes_{\mathbb{Q}} \mathbb{Q}_\ell \hookrightarrow H \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$ by γ_k ($k = 1, 2$) for simplicity.

4 On relation between Eichler-Shimura theory and complex multiplication theory about J_f

In this section we will describe a relation between λ in $f = f_\lambda$ and the homomorphism corresponding to $\alpha_{E/F'}$ in higher dimensional case. The content of this section is essentially stated in the proof of Proposition 1.6 in [5] without detailed proof. We present the results in a slightly different form to be convenient to our purpose.

Firstly we consider Case 1. Then $L := \langle K, H \rangle (\subseteq \mathbb{C})$ is a CM-field with $[L : \mathbb{Q}] = 2n$. We define an isomorphism of \mathbb{Q} -algebras $\iota' : L \xrightarrow{\sim} T = \text{End}_K^0(J_f)$ by

$$a (\in K) \mapsto \kappa(a) (\in Z), \quad x (\in H) \mapsto \theta(x).$$

Then (J_f, ι') is an abelian variety with complex multiplication defined over K in the sense of Shimura (see §19.7 in [6]). Since $\theta(H) \subseteq \text{End}_{\mathbb{Q}}^0(J_f)$, the characteristic polynomial of any element of H acting on $H^0(J_f, \Omega_{J_f/\mathbb{C}}^1) = H^0(J_f, \Omega_{J_f/\mathbb{Q}}^1) \otimes_{\mathbb{Q}} \mathbb{C}$ has \mathbb{Q} -rational coefficients. Therefore, by Lemma 1 in [7] (p.38), the representation of H on $H^0(J_f, \Omega_{J_f/\mathbb{C}}^1)$ is equivalent to the regular representation of H over \mathbb{Q} . It is also proved that Z acts on $H^0(J_f, \Omega_{J_f/\mathbb{C}}^1)$ by scalar multiple. Let $(L, \{\varpi_1, \dots, \varpi_n\})$ be the CM-type of (J_f, ι') , then we have

$$\{\varpi_{1|H}, \dots, \varpi_{n|H}\} = \{\varpi \mid \varpi : H \hookrightarrow \mathbb{C}\}, \quad \varpi_{i|K} = id_K \quad (i = 1, \dots, n)$$

by changing the identification of K as subfield of \mathbb{C} if necessary. Hence the reflex of $(L, \{\varpi_1, \dots, \varpi_n\})$ is $(K, \{id_K\})$. Let $g' : K_{\mathbb{A}}^{\times} \rightarrow L_{\mathbb{A}}^{\times}$ be the canonical map induced from the inclusion $K \subseteq L$. Similarly with case of E/F' , the action of $\text{Gal}(\overline{\mathbb{Q}}/K)$ on $V_{\ell}(J_f)$ determines the homomorphism

$$\delta : \text{Gal}(\overline{\mathbb{Q}}/K) \rightarrow (L \otimes_{\mathbb{Q}} \mathbb{Q}_{\ell})^{\times}$$

and we define $\tilde{\delta} : K_{\mathbb{A}}^{\times} \rightarrow (L \otimes_{\mathbb{Q}} \mathbb{Q}_{\ell})^{\times}$ by the same manner as defining $\tilde{\vartheta}$. The theory of complex multiplication also implies the following:

Corollary 4.1. *For any $x \in K_{\mathbb{A}}^{\times}$, $\tilde{\delta}(x) = (\alpha_{J_f/K}(x)g'(x)^{-1})_{\ell}$, where $\alpha_{J_f/K} : K_{\mathbb{A}}^{\times} \rightarrow L^{\times}$ is the homomorphism corresponding to $\alpha_{E/F'}$ in higher dimensional case.*

Let $\{\mathfrak{p}_1, \dots, \mathfrak{p}_s\}$ be the set of all bad primes of J_f/K . For every \mathfrak{p}_k ($1 \leq k \leq s$), we take the least positive integer t_k such that

$$x \in K_{\mathfrak{p}_k}^{\times} \subseteq K_{\mathbb{A}}^{\times}, \quad x - 1 \in \mathfrak{p}_k^{t_k} \implies \alpha_{J_f/K}(x) = 1.$$

We set $\mathfrak{n} := \mathfrak{p}_1^{t_1} \dots \mathfrak{p}_s^{t_s}$, $G_K(\mathfrak{n}) := \{x \in K_{\mathbb{A}}^{\times} \mid x_{\infty} = 1, x_{\mathfrak{p}_k} = 1 \ (1 \leq k \leq s)\}$, $U_K := \{x \in K_{\mathbb{A}}^{\times} \mid x_{\mathfrak{p}} \in \mathcal{O}_{K_{\mathfrak{p}}}^{\times} \text{ for any finite prime } \mathfrak{p}\}$, and $U_K(\mathfrak{n}) := G_K(\mathfrak{n}) \cap U_K$. We consider the canonical isomorphism $G_K(\mathfrak{n})/U_K(\mathfrak{n}) \xrightarrow{\sim} I_K(\mathfrak{n})$ by which the class represented by $x \in G_K(\mathfrak{n})$ is sent to $il(x) \in I_K(\mathfrak{n})$. Since $U_K(\mathfrak{n}) \subseteq \text{Ker}(\alpha_{J_f/K})$, we obtain the homomorphism

$$\widetilde{\alpha_{J_f/K}} : I_K(\mathfrak{n}) \rightarrow L^{\times}$$

induced from $\alpha_{J_f/K}$. By the two properties of $\alpha_{J_f/K}$: (i) $x \in K_{\infty}^{\times} = \mathbb{C}^{\times} \subseteq K_{\mathbb{A}}^{\times} \implies \alpha_{J_f/K}(x) = 1$; (ii) $x \in K^{\times} \subseteq K_{\mathbb{A}}^{\times} \implies \alpha_{J_f/K}(x) = g'(x) = x$, it holds that

$$\alpha \in K^{\times}, \quad \alpha \equiv 1 \pmod{\mathfrak{n}} \implies \widetilde{\alpha_{J_f/K}}(il(\alpha)) = \alpha.$$

It is clear that $\widetilde{\alpha_{J_f/K}} : I_K(\mathfrak{n}) \longrightarrow L^\times \subseteq \mathbb{C}^\times$ is primitive.

Proposition 4.2. *In Case 1, we have $\lambda = \widetilde{\alpha_{J_f/K}}$ and $\mathfrak{m} = \mathfrak{n}$.*

Next we investigate Case 2. Since J_f is defined over \mathbb{Q} , $\rho_{|K} (\in \text{Gal}(K/\mathbb{Q}))$ acts on $T = \text{End}_K^0(J_f)$. Identifying T with $H \oplus H$ by ε , this action corresponds to the automorphism of $H \oplus H$ defined by $(x, y) \longmapsto (y, x)$. Let ξ_1, ξ_2 be the elements of T which correspond to $(1, 0), (0, 1)$ respectively. We take a positive integer r such that $r\xi_k \in \text{End}_K(J_f)$ ($k = 1, 2$) and set $\xi'_k := r\xi_k$. Then $C := \text{Im}(\xi'_1)$ is an abelian subvariety of J_f defined over K . Since ${}^\rho\xi'_1 = \xi'_2$, we have ${}^\rho C = \text{Im}(\xi'_2)$. So we can define an isogeny $\varphi : J_f \longrightarrow C \times {}^\rho C$ defined over K by $x \longmapsto (\xi'_1(x), \xi'_2(x))$ and this implies $J_f \sim_K C \times {}^\rho C$.

Lemma 4.3. *We have $J_f \sim_{\mathbb{Q}} R_{K/\mathbb{Q}}(C) \sim_{\mathbb{Q}} R_{K/\mathbb{Q}}({}^\rho C)$, where $R_{K/\mathbb{Q}}(C)$ denotes the Weil restriction from K to \mathbb{Q} of C .*

To understand the action of $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on $V_\ell(J_f)$, it is sufficient to do so for that of $\text{Gal}(\overline{\mathbb{Q}}/K)$ on $V_\ell(C)$ by this lemma. Putting $R := \theta^{-1}(\text{End}_{\mathbb{Q}}(J_f))$, we define a ring homomorphism $\iota'' : R \longrightarrow \text{End}_K(C)$ by

$$a \longmapsto (C \ni x \mapsto (\theta(a))(x) \in C)$$

and denote $\iota'' \otimes 1 : H = R \otimes_{\mathbb{Z}} \mathbb{Q} \longrightarrow \text{End}_K^0(C)$ by the same notation ι'' . In Case 2, $K \subseteq H$, so H is a CM-field. Then (C, ι'') is an abelian variety with complex multiplication defined over K . Let H_0 be the maximal real subfield of H and $(H, \{\tau_1, \dots, \tau_{n_1}\})$ ($n_1 := \frac{n}{2}$) be the CM-type of (C, ι'') . Since $H_0 \subseteq \text{End}_K^0(C)$, the characteristic polynomial of any element of H_0 acting on $H^0(C, \Omega_{C/\mathbb{Q}}^1)$ has K -rational coefficients. Since H_0 is totally real, its coefficients also lie in \mathbb{R} . So it has \mathbb{Q} -rational coefficients. It is also proved that $K \subseteq H$ acts on $H^0(C, \Omega_{C/\mathbb{Q}}^1)$ by scalar multiple because $\iota''(K)$ coincides with the center of $\text{End}_{\mathbb{Q}}^0(C) \cong M_{n_1}(K)$. Therefore we have

$$\{\tau_{1|H_0}, \dots, \tau_{n_1|H_0}\} = \{\tau \mid \tau : H_0 \hookrightarrow \mathbb{R}\}, \quad \tau_{i|K} = id_K \quad (i = 1, \dots, n_1)$$

by changing the identification of K as subfield of \mathbb{C} if necessary. Hence the reflex of $(H, \{\tau_1, \dots, \tau_{n_1}\})$ is $(K, \{id_K\})$. Let $g'' : K_{\mathbb{A}}^\times \longrightarrow H_{\mathbb{A}}^\times$ be the canonical map induced from $\gamma_1 : K \hookrightarrow H$. Similary with Case 1, we have

$$\delta' : \text{Gal}(\overline{\mathbb{Q}}/K) \longrightarrow (H \otimes_{\mathbb{Q}} \mathbb{Q}_\ell)^\times, \quad \widetilde{\delta}' : K_{\mathbb{A}}^\times \longrightarrow (H \otimes_{\mathbb{Q}} \mathbb{Q}_\ell)^\times,$$

and the following:

Corollary 4.4. *For any $x \in K_{\mathbb{A}}^\times$, $\widetilde{\delta}'(x) = (\alpha_{C/K}(x)g''(x)^{-1})_\ell$.*

Let n' be the one corresponding to n in case of C/K . Then, as Case 1, we can define

$$\widetilde{\alpha_{C/K}} : I_K(n') \longrightarrow H^\times.$$

Proposition 4.5. *In Case 2, we have $\lambda = \widetilde{\alpha_{C/K}}$ and $m = n'$.*

5 Main results

Let $\beta_{E/F'} : F'_\mathbb{A}^\times \longrightarrow \mathbb{C}^\times$ be the Grössen-character of E/F' . (By definition, $\beta_{E/F'}(x) = (\alpha_{E/F'}(x)N_{F'/K}(x)^{-1})_\infty$.)

Theorem 5.1. *Let E be an elliptic curve with complex multiplication defined over an algebraic number field $F (\subseteq \mathbb{C})$ with $\text{End}_\mathbb{Q}^0(E) \cong K (\subseteq \mathbb{C})$. Put $F' := \langle F, K \rangle (\subseteq \mathbb{C})$. Then the following three conditions are equivalent:*

- (1) E is modular over F .
- (2) There exists a Grössen-character $\gamma : K_\mathbb{A}^\times \longrightarrow \mathbb{C}^\times$ such that $\gamma \circ N_{F'/K} = \beta_{E/F'}$.
- (3) All the points of E of finite order are rational over $\langle F', K_{ab} \rangle = \langle F, K_{ab} \rangle$.

Proof. The equivalence of (2) and (3) is a special case of Theorem 4. p. 511 in [4].

We will prove that (1) implies (2). By assumption, there exists a normalized newform f of weight two (obtained by some $\lambda : I_K(m) \longrightarrow \mathbb{C}^\times$ as $f = f_\lambda$) such that $\text{Hom}_{F'}(E, J_f) \neq \{0\}$. From f we define H as above. Firstly we consider Case 1. We define $\tilde{\chi}_\ell : F'_\mathbb{A}^\times \longrightarrow (T \otimes_\mathbb{Q} \mathbb{Q}_\ell)^\times$ from χ_ℓ by the same manner as defining $\tilde{\vartheta}$ from ϑ in Section 3. By the commutative diagram

$$\begin{array}{ccc} F'_\mathbb{A}^\times & \xrightarrow{\text{norm}} & K_\mathbb{A}^\times \\ \text{reciprocity} \downarrow \text{law} & & \downarrow \text{reciprocity law} \\ \text{Gal}(F'_{ab}/F') & \xrightarrow{\text{restriction}} & \text{Gal}(K_{ab}/K), \end{array}$$

Proposition 2.4, Corollary 3.2, Proposition 3.3, and Corollary 4.1, we have that

$$\tilde{\chi}_\ell(x) = \alpha_{E/F'}(x)^{-1} \alpha_{J_f/K}(N_{F'/K}(x)) \quad \text{for any } x \in F'_\mathbb{A}^\times.$$

(We identify L with T by ι' .) In Case 1, T is a field, so we have

$$\text{Hom}_{F'}(E, J_f) \neq \{0\} \iff \chi = 1 \iff \tilde{\chi}_\ell = 1 \iff \alpha_{J_f/K} \circ N_{F'/K} = \alpha_{E/F'}.$$

We note that Proposition 4.2 is rephrased to that the map

$$G_K(m) \xrightarrow{ii} I_K(m) \xrightarrow{\lambda} \mathbb{C}^\times$$

can be continuously extended to $K_{\mathbb{A}}^{\times}$ by the manner: any $x \in K^{\times} (\subseteq K_{\mathbb{A}}^{\times})$ is mapped to 1 and this extended map, denoted by $\bar{\lambda}$, coincides with $\beta_{J_f/K}$. Then it holds that

$$\alpha_{J_f/K} \circ N_{F'/K} = \alpha_{E/F'} \iff \bar{\lambda} \circ N_{F'/K} = \beta_{E/F'},$$

so we can take $\bar{\lambda}$ as γ in (2).

Next we consider Case 2. By the argument in the proof of Proposition 4.5, the action of $\text{Gal}(\bar{\mathbb{Q}}/F')$ on $V_{\ell}(J_f)$ corresponds to the homomorphism

$$\begin{array}{ccc} \text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) & \xrightarrow{\Phi_{\ell}} & GL_2(H \otimes_{\mathbb{Q}} \mathbb{Q}_{\ell}) \\ \cup & & \cup \\ \text{Gal}(\bar{\mathbb{Q}}/F') & \longrightarrow & (H \otimes_{\mathbb{Q}} \mathbb{Q}_{\ell})^{\times} \oplus (H \otimes_{\mathbb{Q}} \mathbb{Q}_{\ell})^{\times} \\ \cup & & \cup \\ \sigma & \longmapsto & (\delta'(\sigma), \delta'(\rho\sigma\rho)). \end{array}$$

By Proposition 2.4 and Proposition 3.3, we have that one of the following two statements holds:

- (a) $\chi_{\ell}^{(1)}(\sigma) = \gamma_1(\vartheta(\sigma))^{-1}\delta'(\sigma)$, $\chi_{\ell}^{(2)}(\sigma) = \gamma_2(\vartheta(\sigma))^{-1}\delta'(\rho\sigma\rho)$ for any $\sigma \in \text{Gal}(\bar{\mathbb{Q}}/F')$;
- (b) $\chi_{\ell}^{(1)}(\sigma) = \gamma_1(\vartheta(\sigma))^{-1}\delta'(\rho\sigma\rho)$, $\chi_{\ell}^{(2)}(\sigma) = \gamma_2(\vartheta(\sigma))^{-1}\delta'(\sigma)$ for any $\sigma \in \text{Gal}(\bar{\mathbb{Q}}/F')$.

We will prove that (b) is impossible. For this we assume that (b) holds. For $k = 1, 2$, we define $\widetilde{\chi}_{\ell}^{(k)}$ similarly with $\widetilde{\chi}_{\ell}$. If $\sigma|_{F'_{ab}} = [x, F']$ ($[x, F']$ denotes the image of $x \in F'_{\mathbb{A}}^{\times}$ by the reciprocity law of F'), then we have $\rho\sigma\rho|_{K_{ab}} = [\rho(N_{F'/K}(x)), K]$ by the class field theory. Therefore, for any $x \in F'_{\mathbb{A}}^{\times}$, we have

$$\begin{aligned} \widetilde{\chi}_{\ell}^{(1)}(x) &= \gamma_1(\widetilde{\vartheta}(x)^{-1})\widetilde{\delta}'(\rho(N_{F'/K}(x))) \\ &= \gamma_1(\alpha_{E/F'}(x))^{-1}\gamma_1((N_{F'/K}(x))_{\ell})\alpha_{C/K}(\rho(N_{F'/K}(x)))\gamma_1((\rho(N_{F'/K}(x)))_{\ell})^{-1}. \end{aligned}$$

Since $\gamma_1 \circ \rho = \gamma_2$, this is rephrased to that

$$\frac{\gamma_1((N_{F'/K}(x))_{\ell})}{\gamma_2((N_{F'/K}(x))_{\ell})} = \frac{\widetilde{\chi}_{\ell}^{(1)}(x)\alpha_{E/F'}(x)}{\alpha_{C/K}(\rho(N_{F'/K}(x)))}.$$

We can take a transcendental element π of \mathbb{Q}_{ℓ} over \mathbb{Q} and put $x_0 := 1 \otimes 1 + \sqrt{-D} \otimes \pi \in (K \otimes_{\mathbb{Q}} \mathbb{Q}_{\ell})^{\times} \subseteq (F' \otimes_{\mathbb{Q}} \mathbb{Q}_{\ell})^{\times} \subseteq F'_{\mathbb{A}}^{\times}$. Now we suppose that ℓ splits completely in H . Since $K \subseteq H$, we can view $K \subseteq \mathbb{Q}_{\ell}$. By the isomorphism

$$\left(\prod_{\substack{j: H \hookrightarrow \mathbb{Q}_{\ell} \\ j(\sqrt{-D}) = \sqrt{-D}}} j \otimes 1 \right) \oplus \left(\prod_{\substack{r: H \hookrightarrow \mathbb{Q}_{\ell} \\ r(\sqrt{-D}) = -\sqrt{-D}}} r \otimes 1 \right) : H \otimes_{\mathbb{Q}} \mathbb{Q}_{\ell} \xrightarrow{\sim} \underbrace{(\mathbb{Q}_{\ell} \oplus \cdots \oplus \mathbb{Q}_{\ell})}_{\frac{n}{2}} \oplus \underbrace{(\mathbb{Q}_{\ell} \oplus \cdots \oplus \mathbb{Q}_{\ell})}_{\frac{n}{2}},$$

the element

$$\frac{\gamma_1((N_{F'/K}(x_0))_{\ell})}{\gamma_2((N_{F'/K}(x_0))_{\ell})} = \frac{\gamma_1(x_0^d)}{\gamma_2(x_0^d)}$$

is mapped to

$$\left(\left(\frac{1 + \pi\sqrt{-D}}{1 - \pi\sqrt{-D}} \right)^d, \dots, \left(\frac{1 + \pi\sqrt{-D}}{1 - \pi\sqrt{-D}} \right)^d, \left(\frac{1 - \pi\sqrt{-D}}{1 + \pi\sqrt{-D}} \right)^d, \dots, \left(\frac{1 - \pi\sqrt{-D}}{1 + \pi\sqrt{-D}} \right)^d \right),$$

where $d = [F' : K]$. Putting

$$\xi := \frac{\widetilde{\chi_\ell^{(1)}}(x_0) \alpha_{E/F'}(x_0)}{\alpha_{C/K}(\rho(N_{F'/K}(x_0)))} \in H^\times \subseteq (H \otimes_{\mathbb{Q}} \mathbb{Q}_\ell)^\times$$

and taking $j : H \hookrightarrow \mathbb{Q}_\ell$ with $j(\sqrt{-D}) = \sqrt{-D}$, we have that

$$\left(\frac{1 + \pi\sqrt{-D}}{1 - \pi\sqrt{-D}} \right)^d = j(\xi) \text{ in } \mathbb{Q}_\ell.$$

We note that $j(\xi)$ is algebraic over \mathbb{Q} . So we have that

$$\pi = \frac{\sqrt[d]{j(\xi)} - 1}{\sqrt{-D}(1 + \sqrt[d]{j(\xi)})} \in \overline{\mathbb{Q}}.$$

This is a contradiction. Hence we have proved that (a) holds.

We set

$$\iota''' : H \longrightarrow \text{End}_K^0({}^\rho C), \quad a \longmapsto \theta(\rho(a))|_{{}^\rho C}.$$

Then $({}^\rho C, \iota''')$ is an abelian variety with complex multiplication defined over K which has the same CM-type with (C, ι'') . As case of (C, ι'') , we have $\alpha_{{}^\rho C/K} : K_A^\times \longrightarrow H^\times$. Since $\alpha_{{}^\rho C/K} = \rho \circ \alpha_{C/K} \circ \rho$, it holds that

$$(a) \iff \widetilde{\chi_\ell^{(1)}}(x) = \alpha_{E/F'}(x)^{-1} \alpha_{C/K}(N_{F'/K}(x)), \quad \widetilde{\chi_\ell^{(2)}}(x) = \rho(\alpha_{E/F'}(x)^{-1} \alpha_{{}^\rho C/K}(N_{F'/K}(x))) \\ \text{for any } x \in F_A'^\times.$$

Therefore we have

$$\begin{aligned} \text{Hom}_{F'}(E, J_f) \neq \{0\} &\iff \chi^{(1)} = 1 \text{ or } \chi^{(2)} = 1 \iff \chi_\ell^{(1)} = 1 \text{ or } \chi_\ell^{(2)} = 1 \\ &\iff \alpha_{C/K} \circ N_{F'/K} = \alpha_{E/F'} \text{ or } \alpha_{{}^\rho C/K} \circ N_{F'/K} = \alpha_{E/F'}. \end{aligned}$$

Set $\lambda' := \rho \circ \lambda \circ \rho : I_K(\rho(\mathfrak{m})) \longrightarrow \mathbb{C}^\times$. As Case 1, we can construct a Grössen-character $\bar{\lambda}$ (resp. $\bar{\lambda}'$) of K_A^\times from λ (resp. λ'). Then we have

$$\alpha_{C/K} \circ N_{F'/K} = \alpha_{E/F'} \text{ or } \alpha_{{}^\rho C/K} \circ N_{F'/K} = \alpha_{E/F'} \iff \bar{\lambda} \circ N_{F'/K} = \beta_{E/F'} \text{ or } \bar{\lambda}' \circ N_{F'/K} = \beta_{E/F'}.$$

Hence we can take $\bar{\lambda}$ or $\bar{\lambda}'$ as γ in (2).

Finally we will prove that (2) implies (1). By Lemma 2.2, it is sufficient to show that there exists a normalized newform $f = f_\lambda$ of weight two constructed from some $\lambda : I_K(\mathfrak{m}) \longrightarrow \mathbb{C}^\times$ such that $\text{Hom}_{F'}(E, J_f) \neq \{0\}$.

Claim. Let γ be as in (2) and n_0 be the conductor of γ . As defining $\widetilde{\alpha_{J_f/K}}$ from $\alpha_{J_f/K}$ in Section 4, we can also define $\tilde{\gamma} : I_K(n_0) \rightarrow \mathbb{C}^\times$ from γ . Then it holds that for any $x \in K^\times$ s.t. $x \equiv 1 \pmod{n_0}$,

$$\tilde{\gamma}((x)) = x.$$

By Claim, from $\tilde{\gamma}$ we can construct a normalized newform $f = f_{\tilde{\gamma}}$ of weight two. Then the arguments in the proof of the statement: (1) \Rightarrow (2) imply that

$$\begin{aligned} \gamma \circ N_{F'/K} = \beta_{E/F'} &\iff \begin{cases} \alpha_{J_f/K} \circ N_{F'/K} = \alpha_{E/F'} & (\text{if } K \not\subseteq H) \\ \alpha_{G/K} \circ N_{F'/K} = \alpha_{E/F'} & (\text{if } K \subseteq H) \end{cases} \\ &\implies \text{Hom}_{F'}(E, J_f) \neq \{0\}. \end{aligned}$$

So we have proved that (2) \Rightarrow (1). \square

Theorem 5.2. Let E/F , K , F' , and $\beta_{E/F'}$ be as in Theorem 5.1. Assume that the condition (2) in Theorem 5.1 holds. Let \mathfrak{m} be the conductor of γ and set

$$f(z) = f_{\tilde{\gamma}}(z) := \sum_{\substack{\mathfrak{a} \in I_K(\mathfrak{m}) \\ \mathfrak{a}: \text{integral}}} \tilde{\gamma}(\mathfrak{a}) q^{N(\mathfrak{a})} = \sum_{m \geq 1} a_m q^m \quad (q = e^{2\pi iz}).$$

Put $H := \mathbb{Q}(a_m | m \geq 1)$. Then we have the followings:

- (1) For any normalized newform g of weight two, $\text{Hom}_F(E, J_g) \neq \{0\}$ if and only if there exists some γ as above such that $g = f_{\tilde{\gamma}}$.
 (2) Case 1: $K \not\subseteq H$. Then we have

$$J_f \sim_F \underbrace{E \times \cdots \times E}_n \quad (n = \dim J_f = [H : \mathbb{Q}]).$$

Case 2: $K \subseteq H$.

- (a) If $\gamma = \rho \circ \gamma \circ \rho$ on $P := K^\times N_{F'/K}(F'^\times)$, then we have

$$J_f \sim_F \underbrace{E \times \cdots \times E}_n.$$

- (b) If $\gamma \neq \rho \circ \gamma \circ \rho$ on P , then we have that $F = F'$ and there exists an abelian variety A of dimension $\frac{n}{2}$ defined over K such that

$$J_f \sim_F \underbrace{E \times \cdots \times E}_{\frac{n}{2}} \times A_{/F}, \quad \text{Hom}_F(E, A_{/F}) = \{0\}.$$

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